arXiv:dg-ga/9712002v2 4 Dec 1997

POSITIVELY CURVED 7-DIMENSIONAL MANIFOLDS

F. Podestà & L. Verdiani

Abstract. We deal with seven dimensional compact Riemannian manifolds of positive curvature which admit a cohomogeneity one action by a compact Lie group G. We prove that the manifold is diffeomorphic to a sphere if the dimension of the semisimple part of G is bigger than six.

1. Introduction.

Compact Riemannian manifolds of positive curvature have been studied by several authors and, in particular, the homogeneous ones have been classified many years ago by Wallach ([Wa]) and by Berard-Bergery ([BB]). Non homogeneous examples are quite rare and have been found in dimension 6 and 7 by Eschenburg more recently ([Es1], [Es2]). On the other hand it is natural to try to construct new examples on manifolds which are not homogeneous but admit a large group of symmetries; in this context, Searle ([Se]) has classified cohomogeneity one, positively curved Riemannian manifolds in dimension 5 and 6, finding, up to diffeomorphisms, only spheres and complex projective space, while the 4-dimensional case was treated in [HK].

The aim of this note is to study the seven dimensional case, under the assumption of a cohomogeneity one action of a compact Lie group of isometries. More precisely, we say that a compact Lie group G acts on a compact manifold M by cohomogeneity one if it has a hypersurface orbit; in this case, the orbit space M/G is homeomorphic to S^1 or to a closed interval $[0, 1]$. For a detailed exposition and the general results, we refer to [Br],[AA],[AA1].

Our main result is the following:

Theorem. *Let M^7 be a compact, positively curved seven dimensional Riemannian manifold. Let G be a compact Lie group G acting isometrically and almost faithfully on M^7 by cohomogeneity one. If the semisimple part of G has dimension bigger than 6, then M^7 is diffeomorphic to a sphere S^7.*

Remark. Actually, we prove that, if we want to find a positively curved, cohomogeneity one manifold not diffeomorphic to S^7, then the only "candidate" group is $G = SU(2) \times SU(2)$. This last case is much more complicated to be handled with and will be object of a forthcoming paper.

1991 *Mathematics Subject Classification.* 53C21, 57S15.

Typeset by $\mathcal{AMS}$-TEX

It is worthy noting here that, for istance, the Berger space $V^7 = \frac{Sp(2)}{SU(2)}$ (see [Be]), endowed with the normal homogeneous metric of positive curvature, admits an isometric G-action of cohomogeneity one, where $G = Sp(1) \times Sp(1) \subset Sp(2)$. We recall that the space V^7 is obtained by considering the immersion of $SU(2)$ into $Sp(2)$ corresponding to the (essentialy unique) irreducible representation ρ of $SU(2)$ on $\mathbb{C}^4 \cong \mathbb{H}^2$, which is of quaternionic type; the space V is not homeomorphic to S^7, although it has the same real cohomology (see [Be]).

In order to prove that the action of G on V^7 has cohomogeneity one, we observe the following. If $X \in \mathfrak{a}_1$, the Lie algebra of $SU(2)$, then $\rho(X)$ leaves exactly two quaternionc lines invariant, which we call $Q_1(X), Q_2(X)$ and the two restricted representations $\rho(X)|_{Q_i(X)}$ are not equivalent. It then follows that, for generic $g \in Sp(2)$, the intersection $G \cap gSU(2)g^{-1}$ will be finite and a principal G-orbit has dimension 6.

2. Proof of the main Theorem.

First of all we recall that the manifold M^7, being of positive curvature and compact, has finite fundamental group, so that we shall always suppose that M^7 is simply connected. This implies that there is no fibration of M^7 on S^1, so that there are exactly two singular orbits. Throughout the following, we will use the symbols K to denote a fixed regular isotropy subgroup and H, H' to denote two singular isotropy subgroups with $K \subset H \cap H'$; it follows from the general theory that H/K and H'/K are diffeomorphic to spheres of positive dimensions, since there cannot be execeptional orbits (see [Br], [AA]).

We shall use the following results ([GS]):

Theorem 1.1, [GS]. *Let (M^n, g) be a simply connected, n-dimensional compact CPCM.*

1. *If a torus T^k acts isometrically on M, then $k \leq [\frac{n+1}{2}]$, with equality if and only if M^n is diffeomorphic to a sphere S^n or a complex projective space $\mathbb{CP}^{n/2}$.*
2. *If T^1 or $SU(2)$ acts isometrically on M^n with a fixed point set of codimension two or less than four respectively, then M^n is diffeomorphic to a sphere S^n, to a complex projective space $\mathbb{CP}^{n/2}$ or to a quaternionic projective space $\mathbb{HP}^{n/4}$.*

So, we will suppose that the group G has rank less or equal to 3. Furthermore, since the regular orbit has dimension 6, the group G must have rank at least 2.

We will divide our analysis according to the difference d between the rank of G and the rank of the regular isotropy subgroup K. Moreover we note that, since the group G acts almost effectively on M, it acts almost effectively on the regular orbit too, so that K cannot contain any ideal of G.

Case $d = 0$.

In this case, K has maximal rank and therefore G is semisimple. We will now subdivide our study according to the rank of G:

Subcase rank(G) = 2.

If G is not simple, then G is locally isomorphic to $SU(2) \times SU(2)$ and K must be 2-dimensional, so that G/K has dimension 4. So, we are left with the case G simple. Now, all compact rank 2, simple Lie groups are locally isomorphic to $SU(3), Spin(5), G_2$ (recall that $Sp(2) = Spin(5)$). We examine each case separately:

a) If $G = G_2$, then K has dimension 8 and has maximal rank. Since the maximal subalgebras of maximal rank of $\mathfrak{g}_2$ are $\mathfrak{a}_2$ and $2\mathfrak{a}_1$ (see [GG]), we see that the Lie algebra $\mathfrak{k}$ of K must be maximal, isomorphic to $\mathfrak{a}_2$. In this case, the Lie algebra of a singular isotropy subgroup must coincide with $\mathfrak{g}$, since there are no exceptional orbits. It then follows that the action of G has exactly two fixed points and the manifold is diffeomorphic to S^7.

b) If $G = Spin(5)$, then K has dimension 4 and rank 2; so the only possibility for $\mathfrak{k}$ is $\mathfrak{k} \cong \mathbb{R} + \mathfrak{a}_1$. Again the Lie algebra $\mathfrak{k}$ is maximal and G should have a fixed point by the same argument as above. But $Spin(5)$ does not act transitively on a 6-dimensional sphere (see [AA]) and this case is ruled out.

c) If $G = SU(3)$, then K has dimension 2 and rank 2, so that K^o, the connected component of K, coincides with a maximal torus T^2 of G. We fix K once for all. We now consider a singular isotropy subgroup H, with Lie algebra $\mathfrak{h}$: now H contains K and H/K is diffeomrophic to a sphere. It is not difficult to see that the only possibility for $\mathfrak{h}$ is $\mathfrak{h} \cong \mathbb{R} + \mathfrak{a}_1$, maximal Lie subalgebra of maximal rank in $\mathfrak{a}_2 = \mathfrak{g}$. We observe that, each singular orbit is of codimension 4 in M^7, so that it is simply connected; therefore H is connected, isomorphic to $U(2)$ and K is also connected, isomorphic to T^2. There are exactly three immersions of $U(2)$ into $SU(3)$, containing the maximal torus T^2 and they are mutually conjugate by the Weyl group. It then follows that we have to consider exactly two cases: the first when the two singular isotropy subgroups H, H' are isomorphic to $U(2)$ but with different immersions and the second one, when $H = H'$.

Lemma 3.1. *Given the triple (H, K, H') of subgroups of $G = SU(3)$ with $K = T^2$, maximal torus and $H, H' \cong U(2)$, then*

(1) *if $H \neq H'$, then the manifold is diffeomorphic to S^7;*
(2) *if $H = H'$, then the manifold does not carry any positively curved, G-invariant metric.*

Proof. The first case is easily handled, since the 7-dimensional sphere admits a cohomogeneity one action of the group $SU(3)$, induced by the adjoint representation, which admits a triple of subgroups as in (1).

In case $H = H'$, we decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3$, where $\mathfrak{m}_i,\ i = 1, 2, 3$ are two-dimensional, irreducible and mutually inequivalent $\mathfrak{k}$-modules. We can fix $\mathfrak{h} = \mathfrak{h}' = \mathfrak{k} + \mathfrak{m}_1$. We now fix a non zero vector $v \in \mathfrak{m}_2$ and consider a normal geodesic $\gamma : \mathbb{R} \to M$ w.r.t. a positively curved G-invariant metric g on M; we choose γ so that it induces the triple $\theta = (H, K, H')$.

First of all, we claim that the Killing vector field X induced by v on M never vanishes along γ. This is clear since $\mathfrak{h} = \mathfrak{h}'$ and $v \notin \mathfrak{h}$.

We now consider the smooth function $f(t) = ||X||_{\gamma(t)}$ for $t \in \mathbb{R}$ and we claim that f is a concave positive function, which is not possible. This will conclude our proof.

It will be enough to check that $f"(t) < 0$ for all t such that $\gamma(t)$ is a regular point. First, we observe that the tangent space to a regular orbit splits into K-irreducible, mutually inequivalent submodules, so that the shape operator of the regular orbit hypersurface will preserve each submodule and will be a multiple of the identity operator on $\mathfrak{m}_2$. Therefore, if we denote by D the Levi-Civita connection of g, we have that $D_{\gamma(t)'}X$ is a multiple of $X_{\gamma(t)}$; we then have

$$\begin{aligned} 2R_{X\gamma' X\gamma'} &= 2||D_{\gamma'}X||^2 - \frac{d^2}{dt^2} f^2 \\ &= 2\frac{g(D_{\gamma'}X, X)^2}{f^2} - 2(f')^2 - 2ff" = -2ff" > 0, \end{aligned},$$

since $g(D_{\gamma'}X, X) = ff'$. $\square$

We now proceed considering the next

Subcase rank(G) = 3.

We have to distinguish G simple or not simple.

Let us suppose that G is not simple: then G is locally isomorphic to either $G \cong SU(2)^3$ or $G = SU(2) \times G_1$ with G_1 simple of rank 2.

If $G \cong SU(2)^3$, then K^o must be of the form $K^o = (T^1)^3$, where each T^1 is a maximal torus in $SU(2)$. But then a singular isotropy subgroup H should have a connected component equal to $SU(2) \times (T^1)^2$; both singular orbits G/H and G/H' would then be in codimension 3, hence they would be simply connected. Moreover, since the G-action on each singular orbit is not faithful, they would be totally geodesic (see [PV]) and diffeomorphic to $S^2 \times S^2$; this is not possible, because $S^2 \times S^2$ does not carry any homogeneous positively curved metric (see [HK]).

If $G = SU(2) \times G_1$, where G_1 is simple of rank 2, then K^o is of the form $K^o = T^1 \times K_1$, where $K_1 \subset G_1$ subgroups of maximal rank; since $\dim G/K = 6$, we have that $\dim G_1/K_1 = 4$ and hence $\dim G_1 \leq 10$. Now G_1 can be either $SU(3)$ or $Spin(5)$.

If $G_1 = SU(3)$, then $G = SU(2) \times SU(3)$ with $K^o = T^1 \times U(2)$ and since $U(2)$ is maximal in G_1, any singular isotopy subgroup $H \supset K$ must be of the form $H^o = SU(2) \times U(2)$; so any singular orbit is totally geodesic (see [PV]) and of codimension 3, contradicting Frankel Theorem (see [Fr]).

If $G_1 = Spin(5)$, then $\dim \mathfrak{k}_1 = 6$: looking at the list of all maximal subalgebras of maximal rank in $\mathfrak{so}(5)$, we see that $\mathfrak{k}_1$ is maximal and siomorphic to $2\mathfrak{a}_1$. Again, the same argument as above rules this case out.

We are left with the case where G is simple of rank 3. Then G is locally isomorphic to $SU(4), Spin(7), Sp(3)$. We have that $\dim K = \dim G - 6$ and $Sp(3)$ does not have any such subgroup.

In case $G = SU(4)$, we have that $\dim K = 9$. But a 9-dimensional, rank 3 subalgebra of $\mathfrak{a}_3$ must be maximal, isomorphic to $\mathfrak{a}_2 + \mathbb{R}$; this means that any

singular isotropy subalgebra must coincide with $\mathfrak{g}$. But in this case G/K is not diffeomorphic to a sphere, so it is impossible.

In case $G = Spin(7)$, the same kind of arguments show that $K^o = Spin(6)$, maximal subalgebra; then G must have exactly two fixed points and M is diffeomorphic to S^7.

Case $d = 1$**.**

We subdivide this case into two subcases, according to the rank of G equal to 2 o 3.
(a) If rank(G)=2, then the rank of K is 1, hence $K^o \cong T^1$ or $SU(2)$. Therefore $\dim G = 7$ or 9. But there is no compact group G of rank 2 and dimension 7 or 9.
(b) If the rank of G is 3, then K^o belongs to the list:

$$\{T^2, T^1 \times SU(2), SU(2)^2, SU(3), Spin(5), G_2\} \ .$$

It then follows that $\dim K$ belongs to the set $\{2, 4, 6, 8, 10, 14\}$ and therefore $\dim G \in \{8, 10, 12, 14, 16, 20\}$. Now it easy to see that there is no compact group of rank 3 and with the indicated dimension.

Case $d = 2$**.**

If the rank of G is 2, then K is discrete and G has dimension 6, hence G is locally isomorphic to $SU(2)^2$.

If the rank of G is 3, then K^o is either T^1 or $SU(2)$. Therefore $\dim G \in \{7, 9\}$. So G cannot be simple. If G is semisimple, not simple, then the only possibility is $G \cong SU(2)^3$, while, if G is not semisimple, then we have two possibilities, namely $G \cong T^1 \times SU(2)^2$ or $T^1 \times SU(3)$.

Summing up, we found that in cases $d = 0$ and $d = 1$, the manifold M must be diffeomorphic to the sphere S^7. If $d = 2$, then we have the following possibilities for the pair (G, K^o):

$n.$	G	K^o
1	$T^1 \times SU(3)$	$SU(2)$
2	$SU(2)^3$	$SU(2)$
3	$T^1 \times SU(2)^2$	T^1
4	$SU(2)^2$	$\{1\}$

Table 1.

We now prove the following

Lemma 2.2. *Case (1) in Table 1 occurs only if M is diffeomorphic to S^7.*

Proof. First of all we have to identify the subgroup $K^o \cong SU(2)$ inside G. Note that there are, up to conjugation, exactly two immersions of $SU(2)$ into $SU(3)$, corresponding to an irreducible or reducible representation of $SU(2)$ on $\mathbb{C}^3$. We

want to prove that $SU(2)$ inside $SU(3)$ must correspond to a reducible representation. Indeed, if not, we consider the Lie algebra $\mathfrak{h}$ of a singular isotropy subgroup $H \supset K$: since H/K must be diffeomorphic to a sphere and $\mathfrak{su}(2)$ acts irreducibly on $\mathbb{C}^3$, then the only possibility for $\mathfrak{h}$ is $\mathfrak{h} \cong \mathfrak{su}(2) + \mathbb{R}$. Moreover, since $\mathfrak{k}$ has trivial centralizer in $\mathfrak{su}(3)$, we get that $\mathfrak{h}$ contains the center of $\mathfrak{g}$. So, the action of G on G/H is not faithful and any singular orbit is totally geodesic in codimension 2; but this contradicts Frankel Theorem (see [Fr]).

We then have that the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$ has real dimension 2 and therefore $K^o = SU(2)$ fixes 3 dimensions; but then by Theorem 1.1, the manifold is diffeomorphic to S^7. $\square$

Lemma 2.3. *Case (2) in Table 1 occurs only if M is diffeomorphic to S^7.*

Proof. Since the Lie algebra $\mathfrak{k}$ cannot contain ideals of $\mathfrak{g}$, we have only two possibilities, namely (i): $\mathfrak{k} = \mathfrak{su}(2)^{\Delta} \subset 2\mathfrak{su}(2)$ or (ii): $\mathfrak{k} = \mathfrak{su}(2)^{\Delta} \subset 3\mathfrak{su}(2)$, where the symbol Δ means diagonal embedding.

In case (i), $\mathfrak{k}$ fixes exactly four dimension and Theorem 1.1 implies the claim.

In case (ii), we consider a singular isotropy subalgebra $\mathfrak{h}$; since $\mathfrak{k}$ has trivial centralizer in $\mathfrak{g}$, we see that $\mathfrak{h} \cong \mathfrak{su}(2) + \mathfrak{su}(2)$ containing $\mathfrak{k}$ diagonally. But then, $\mathfrak{h}$ must contain an ideal $\mathfrak{su}(2)$ of $\mathfrak{g}$; if we call $\mathfrak{p}$ such an ideal, we see that the Lie group $P = SU(2)$ corresponding to $\mathfrak{p}$ fixes pointwise the orbit G/H in codimension four and again by Thm. 1.1 we get our claim. $\square$

We now want to analyze the case (3) carefully. In this case $G = T^1 \times SU(2)^2$ and the connected component $K^o = T^1$. We look for possible singular isotropy subalgebras $\mathfrak{h} \supset \mathfrak{k}$: if $\mathfrak{n}$ denotes the kernel of the slice representation of $\mathfrak{h}$, then we have the following possibilities: if $\mathfrak{n} = \mathfrak{k}$, then $\mathfrak{h}$ can be isomorphic to $(i) : \mathbb{R} + \mathfrak{su}(2)$ or $(ii) : \mathbb{R}^2$ and if $\mathfrak{n}$ is trivial, then $\mathfrak{h}$ can be isomorphic to $(iii) : \mathbb{R} + \mathfrak{su}(2)$ or $(iv) : \mathfrak{su}(2)$.

We now observe the following:

(1) the Lie algebra $\mathfrak{k}$ can be supposed to have non trivial projection onto each $\mathfrak{su}(2)$-factors; otherwise it is easy to show that $\mathfrak{k}$ would fix 5 dimensions and we could apply Thm.1.1. It then follows that the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$ is abelian and this excludes the possibility (i) for $\mathfrak{h}$.

(2) If $\mathfrak{h}$ is isomorphic to $\mathbb{R} + \mathfrak{su}(2)$, then the semisimple part of $\mathfrak{h}$ can be supposed to be immersed diagonally, otherwise it would fix three dimensions and again we could apply Thm 1.1. So the center of $\mathfrak{h}$ would coincide with the center of $\mathfrak{g}$; then we could restrict the action of G to its semisimple part $G^s = SU(2)^2$, which would still act transitively on the singular orbit G/H and the semisimple part $\mathfrak{h}^s$ would act by cohomogeneity one on the normal space to G/H. Therefore we would reduce to case (1) in Table 1.

(3) If $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$, then again we can suppose it to be diagonally embedded. The tangent space of G/H at some point p is an $\mathfrak{su}(2)$-module and its complexification splits, as $\mathfrak{h}$-module, as $\mathbb{C} + S^2(\mathbb{C}^2)$. Now, the second fundamental form h of G/H at p gives rise to an $\mathfrak{su}(2)$-fixed vector in the

space $S^2(\mathbb{C}+S^2(\mathbb{C}^2))\otimes S^2(\mathbb{C}^2)$, which is easily seen to have no such vector. Therefore, G/H is totally geodesic and finitely covered by $S^1\times S^3$: this is impossible, since, being positively curved, it should have finite π_1.

It the follows that both singular isotropy subalgebras can be supposed to be isomorphic to $\mathbb{R}^2$. It is also clear that if $\mathfrak{h}$ or $\mathfrak{h}'$ contains the center of $\mathfrak{g}$, then we could apply Thm 1.1; therefore we can suppose that neither $\mathfrak{h}$ nor $\mathfrak{h}'$ contains the center.

We now consider the decomposition $\mathfrak{g}=\mathfrak{h}+\mathbb{R}+\mathfrak{m}_1+\mathfrak{m}_2$, where $\mathbb{R}$ is a trivial $\mathfrak{h}$-module and $\mathfrak{m}_i,\ i=1,2$ are irreducible $\mathfrak{h}$-modules (note that we are supposing that $\mathfrak{h}$ does not contain the center and that $\mathfrak{k}$ is embedded diagonally) of real dimension two.

It is easy to see that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are $\mathfrak{h}$-inequivalent modules, otherwise $\mathfrak{h}$ would contain the center of $\mathfrak{g}$. We now consider the second fundamental form h of G/H: since $\mathfrak{h}$ acts trivially on $\mathbb{R}$, it follows that $\mathbb{R}\subset\ker h$. Moreover we consider the kernels $\mathfrak{n}_i,\ i=1,2$ of the actions of $\mathfrak{h}$ on $\mathfrak{m}_i$: it is clear that each $\mathfrak{n}_i$ is not equal to $\mathfrak{k}$ and therefore it acts transitively on the normal space. It follows that $h|_{\mathfrak{m}_i\times\mathfrak{m}_i}=0$. Moreover $\mathfrak{n}_1$ does not act trivially on $\mathfrak{m}_2$ (otherwise $\mathfrak{n}_1$ should coincide with the center of $\mathfrak{g}$, which is not contained in $\mathfrak{g}$): therefore, by the invariance of h under the $\mathfrak{h}$-action, we see that, if $h|_{\mathfrak{m}_1\times\mathfrak{m}_2}\neq 0$, then $\mathfrak{m}_2$ is equivalent to V^*, where V denotes the normal space to G/H. The same argument with $\mathfrak{n}_2$, shows then that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are equivalent, a contradiction. Therefore $h=0$. But then G/H is a positively curved, homogeneous manifold of dimension 5, hence finitely covered by S^5; on the other hand the group G cannot act transitively on a 5-dimensional sphere (see e.g. [AA]).

We have therefore proved that case (3) can be reduced to case (4) in Table 1, if we want to discard manifolds which are diffeomorphic to spheres. This concludes the proof of our main theorem. $\square$

References

[AA] A.V. Alekseevsky and D.V.Alekseevsky, *G-manifolds with one dimensional orbit space*, Adv. in Sov. Math. **8** (1992), 1–31.

[AA1] ———, *Riemannian G-manifolds with one dimensional orbit space*, Ann. Glob. Anal. and Geom. **11** (1993), 197–211.

[BB] L. Berard-Bergery, *Les variétés Riemanniennes homogènes simplement connexes de dimension impaire à courbure strictement positive*, J. Math. Pures Appl. **55** (1976), 47–68.

[Be] M. Berger, *Les variétés Riemanniennes homogènes normal simplement connexes à courbure strictement positive*, Ann. Scuola Normale Superiore, Pisa **15** (1961), 179–246.

[Br] G.E. Bredon, *Introduction to compact transformation groups*, Acad. Press N.Y. London, 1972.

[Es1] J.H. Eschenburg, *New examples of manifolds with strictly positive curvature*, Invent. Math. **66** (1982), 469–480.

[Es2] ———, *Inhomogeneous spaces of positive curvature*, Diff. Geom. and Its Appl. **2** (1992), 123–132.

[Fr] T.T Frankel, *Manifolds of positive curvature*, Pacific J. Math. **61** (1961), 165–174.

[GG] M. Goto, F.D. Grosshans, *Semisimple Lie Algebras*, vol. 38, Lecture Notes in Pure and Appl. Math., Marcel Dekker, 1978.

[GS] K. Grove and C. Searle, *Positively curved manifolds with maximal symmetry-rank*, J. Pure Appl. Algebra **91** (1994), 137–142.

[HK] W.Y. Hsiang and B. Kleiner, *On the topology of positively curved 4-manifolds with symmetry*, J. Diff. Geometry **30** (1989), 615–621.

[PV] F. Podestà, L. Verdiani, *Totally geodesic orbits of isometries* (1997), preprint.

[Se] C. Searle, *Cohomogeneity and positive curvature in low dimension*, Math.Z. **214** (1993), 491–498.

[Wa] N.R. Wallach, *Compact Homogeneous manifolds with strictly positive curvature*, Ann. of Math. **96** (1972), 277–295.

University of Florence, Ist. Matematico and Dip. di Matematica "U.Dini", Italy
E-mail address: `podesta@ udini.math.unifi.it;  verdiani@ udini.math.unifi.it`